AD NUMBER
AD085982
LIMITATION CHANGES
TO: **Approved for public release; distribution is unlimited. Document partially illegible.**
FROM: **Distribution authorized to U.S. Gov't. agencies and their contractors; Administrative/Operational Use; AUG 1954. Other requests shall be referred to Air Force Research and Development Command, Project Rand, Washington, DC 20330. Document partially illegible.**
AUTHORITY
rand ltr via AFRDC, 6 Sep 1966

UNCLASSIFIED

AD NUMBER
AD085982
CLASSIFICATION CHANGES
TO: UNCLASSIFIED
FROM: CONFIDENTIAL
AUTHORITY
31 aug 1966, DoDD 5200.10 gp-4

THIS PAGE IS UNCLASSIFIED

AD 85982

Armed Services Technical Information Agency

Reproduced by

DOCUMENT SERVICE CENTER

KNOTT BUILDING, DAYTON, 2, OHIO

254932

C O N F I D E N T I A L

U.S. AIR FORCE

Project RAND

RESEARCH MEMORANDUM

This material contains information affecting the national defense of the United States within the meaning of the espionage laws, Title 18 U.S.C., Secs. 793 and 794, the transmission or the revelation of which in any manner to an unauthorized person is prohibited by law.

SEE TITLE PAGE FOR DISTRIBUTION RESTRICTIONS

The RAND Corporation

SANTA MONICA · CALIFORNIA

C O N F I D E N T I A L

CONFIDENTIAL

U.S. AIR FORCE
Project RAND
RESEARCH MEMORANDUM

This material contains information affecting the national defense of the United States within the meaning of the espionage laws, Title 18 U.S.C., Secs. 793 and 794, the transmission or the revelation of which in any manner to an unauthorized person is prohibited by law.

SEE TITLE PAGE FOR DISTRIBUTION RESTRICTIONS

This is a working paper. Because it may be expanded, modified, or withdrawn at any time, permission to quote or reproduce must be obtained from RAND. The views, conclusions, and recommendations expressed herein do not necessarily reflect the official views or policies of the U. S. Air Force.

The RAND Corporation
SANTA MONICA · CALIFORNIA

CONFIDENTIAL

CONFIDENTIAL

PROJECT RAND

RESEARCH MEMORANDUM

(Uncl) STATISTICAL THEORY OF NAVIGATION EMPLOYING INDEPENDENT INERTIAL AND VELOCITY MEASUREMENTS: MINIMUM RMS ERROR IN COMPUTED POSITION

P. Swerling
E. Reich

RM-1321

17 August 1954

Copy No. 46

DISTRIBUTION RESTRICTIONS

No restrictions on further distribution, other than those imposed by security regulations.

The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

CONFIDENTIAL

Correction to RM-1321: Statistical Theory of Navigation Employing Independent Inertial and Velocity Measurements: Minimum RMS Error in Computed Position

by P. Swerling and E. Reich

The fourth line of Eq. (II.5), p. 4, should read:

$$\Delta = L_{11}L_{22} - L_{12}^2$$

SUMMARY

Analysis of navigation systems employing independent inertial and velocity measurements, begun in Ref. 1, is continued. Explicit formulas are given for minimum rms error in computed position as a function of time of flight. Curves based on these formulas are presented, showing the results for a number of illustrative cases.

TABLE OF CONTENTS

LIST OF SYMBOLS

Symbol	Definition	Page First Used
$x(t)$	vehicle position at time t	1
$\hat{x}(t;T)$	optimum computer's estimate of $x(t)$ based on all the dial readings up to time T	1
$\hat{x}(T)$	$\hat{x}(T;T)$	1
T	elapsed time since beginning of flight	1
$\gamma_1, \gamma_2, \gamma_3, \gamma_4, \alpha_2, \beta_2, \alpha_1, \beta_1$	parameters describing instrument error statistics	1,2,5,16
$\mathcal{E}_{\hat{x}}(t;T)$	$\hat{x}(t;T) - x(t)$	2
$\mathcal{E}_{\hat{x}(T)}$	$\hat{x}(T;T) - x(T)$	2
ξ_{ij}	defined by Eq. (II.1)	2
η_{ij}	defined by Eq. (III.1)	15
$\phi_B(s,t)$	accelerometer error autocorrelation function	1
$\phi_V(s,t)$	velocity dial error autocorrelation function	1
Ω	frequency of earth's radius pendulum	3
μ	defined by Eq. (II.4)	4

I. INTRODUCTION

This report is an extension of the work begun in Ref. 1. This is not a self-contained report; rather, a reading of Ref. 1 is a necessary basis for the understanding of the following material. Repetition of expository material and results obtained in Ref. 1 will be kept to a minimum.

The major results of Ref. 1 were obtained for two cases, viz:

<u>Case 1</u> $\phi_B(s,t) = \gamma_1$

<u>Case 2</u> $\phi_V(s,t) = \gamma_2$

where $\phi_B(s,t)$ and $\phi_V(s,t)$ are the autocorrelation functions of the accelerometer dial error and the velocity dial error, respectively. Formulas were derived for the optimum method of position computation in each case; also, formulas were given by means of which the rms error in computed position as a function of time of flight could be derived.

In the following pages, the actual formulas for rms error in computed position will be given; also, computations based on these formulas will be given for a number of illustrative cases.

As in Ref. 1, let, for $0 \leq t \leq T$,

$x(t)$ = true position at time t

$\hat{x}(t;T)$ = optimum computer's estimate of $x(t)$, based on all the dial readings up to time T.

Denote $\hat{x}(T;T)$ simply by $\hat{x}(T)$.

Also let

$$(I.1)\qquad \begin{aligned} \epsilon_{\hat{x}}(t;T) &= \hat{x}(t;T) - x(t) \\ \epsilon_{\hat{x}(T)} &= \epsilon_{\hat{x}}(T;T) = \hat{x}(T) - x(T) \end{aligned}$$

and

$$(I.2)\qquad \overline{\epsilon^2_{\hat{x}}(t;T)} = \overline{\left[x(t;T) - x(t)\right]^2}$$

the mean being taken over an ensemble of flights.

As in Ref. 1, the following notation is used for description of the dial error statistics:

$\epsilon_B(t)$ = accelerometer dial error

$\epsilon_V(t)$ = velocity dial error

ϵ_{X_0} = independent initial position dial error

$\epsilon_{X'_0}$ = independent initial velocity dial error

The ensemble means of all these errors are zero; also

$$(I.3)\qquad \overline{\epsilon^2_{X_0}} = \gamma_3 \; ; \quad \overline{\epsilon^2_{X'_0}} = \gamma_4$$

and

$$(I.4)\qquad \begin{aligned} \phi_B(s,t) &= \overline{\epsilon_B(s)\,\epsilon_B(t)} \\ \phi_V(s,t) &= \overline{\epsilon_V(s)\,\epsilon_V(t)} \end{aligned}$$

(All averages are taken over an ensemble of flights.)

II. CASE 1: $\phi_B(s,t) = \gamma_1$

Let $0 = t_0 < t_1 < \cdots < t_{n-1} = T$ be n equally spaced time points in the interval (0,T). Let

(II.1)
$$\left[\xi_{ij}\right] = \left[\phi_V(t_i, t_j)\right]^{-1} \qquad \text{(matrix inverse)}$$

Also let

$$\left[A_{11}\right]_n = \Omega^2 \sum_{i,j=0}^{n-1} \xi_{ij} \sin \Omega t_i \sin \Omega t_j$$

(II.2)
$$\left[A_{12}\right]_n = \Omega \sum_{i,j=0}^{n-1} \xi_{ij} \sin \Omega t_i \cos \Omega t_j$$

$$\left[A_{22}\right]_n = \sum_{i,j=0}^{n-1} \xi_{ij} \cos \Omega t_i \cos \Omega t_j$$

and

$$A_{11} = \lim_{n \to \infty} \left[A_{11}\right]_n$$

(II.3)
$$A_{12} = \lim_{n \to \infty} \left[A_{12}\right]_n$$

$$A_{22} = \lim_{n \to \infty} \left[A_{22}\right]_n$$

The quantities A_{11}, A_{12}, A_{22} are functions of T.

Reference 1 gave formulas (Eqs. III.16, III.17', III.20 of Ref. 1) from which could be derived the formula for $\overline{\mathcal{E}^2_{\hat{x}}(t;T)}$. The result can be expressed as follows:

Let

$$\mu = \frac{\frac{\gamma_1}{\Omega^4} \cdot \gamma_3}{\frac{\gamma_1}{\Omega^4} + \gamma_3} \tag{II.4}$$

and

$$\left\{ \begin{aligned} L_{11} &= A_{11} + \frac{1}{\frac{\gamma_1}{\Omega^4} + \gamma_3} \\ L_{12} &= A_{12} \\ L_{22} &= A_{22} + \frac{1}{\gamma_4} \\ \Delta &= L_{11}\, L_{12} - L_{12}^2 \end{aligned} \right. \tag{II.5}$$

Then, for $0 \leq t \leq T$,

$$\overline{\mathcal{E}^2_{\hat{x}}(t;T)} = \mu + \frac{1}{\Delta} \left\{ L_{22} \left(\cos \Omega t - \frac{\mu}{\gamma_3} \right)^2 + 2L_{12} \left(\cos \Omega t - \frac{\mu}{\gamma_3} \right) \frac{\sin \Omega t}{\Omega} + L_{11} \frac{\sin^2 \Omega t}{\Omega^2} \right\} \tag{II.6}$$

The quantity $\overline{\mathcal{E}^2_{\hat{x}(T)}} = \overline{\left[\hat{x}(T) - x(T)\right]^2}$ is equal to $\overline{\mathcal{E}^2_{x}(T;T)}$. Thus

$$\text{(II.6')} \qquad \overline{\mathcal{E}^2_{\hat{x}(T)}} = \mu \cdot \frac{1}{3}\left\{ L_{22}\left(\cos \Omega T - \frac{\mu}{\gamma_3}\right)^2 + 2L_{12}\left(\cos \Omega T - \frac{\mu}{\gamma_3}\right)\frac{\sin \Omega T}{\Omega} + L_{11}\frac{\sin^2 \Omega T}{\Omega^2}\right\}$$

These formulas hold for any $\phi_V(s,t)$.

In Ref. 1, explicit expressions were obtained for A_{11}, A_{12}, A_{22} for the case:

$$\text{(II.7)} \qquad \phi_V(s,t) = \gamma_2 + \beta_2 e^{-\alpha_2 |s-t|}$$

The results were

$$\text{(II.81)} \qquad A_{11} = \frac{\Omega^2}{2\beta_2}\left\{ \frac{\alpha_2 T}{2}\left(1 + \frac{\Omega^2}{\alpha_2^2}\right) - \frac{\alpha_2}{4\Omega}\left(1 - \frac{\Omega^2}{\alpha_2^2}\right)\sin 2\Omega T + \sin^2 \Omega T - \frac{\left[\frac{\alpha_2}{\Omega}(1 - \cos \Omega T) + \sin \Omega T\right]^2}{2 + \alpha_2 T + \frac{2\beta_2}{\gamma_2}}\right\}$$

(II.8ii)

$$A_{12} = \frac{1}{2\beta_2}\left\{\frac{\alpha_2}{4\Omega}\left(1 - \frac{\Omega^2}{\alpha_2^2}\right)\sin^2 \Omega T + \frac{1}{2}\sin 2\Omega T\right.$$

$$\left. - \frac{\left[\frac{\alpha_2}{\Omega}(1 - \cos \Omega T) + \sin \Omega T\right]\left[\frac{\alpha_2}{\Omega}\sin \Omega T + 1 + \cos \Omega T\right]}{2 + \alpha_2 T + \frac{2\beta_2}{\gamma_2}}\right\}$$

(II.8iii)

$$A_{22} = \frac{1}{2\beta_2}\left\{\frac{\alpha_2 T}{2}\left(1 + \frac{\Omega^2}{\alpha_2^2}\right) + \frac{\alpha_2}{4\Omega}\left(1 - \frac{\Omega^2}{\alpha_2^2}\right)\sin 2\Omega T + 1 + \cos^2 \Omega T\right.$$

$$\left. - \frac{\left[\frac{\alpha_2}{\Omega}\sin \Omega T + 1 + \cos \Omega T\right]^2}{2 + \alpha_2 T + \frac{2\beta_2}{\gamma_2}}\right\}$$

In Figs. 1-7 are shown curves of the quantity $\left[\overline{\mathcal{E}^2_{\hat{x}(T)}}\right]^{1/2}$ as a function of T for $\phi_V(s,t) = \gamma_2 + \beta_2 e^{-\alpha_2|s-t|}$. Each curve is determined by specification of the six statistical parameters $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \alpha_2, \beta_2$.

The following table gives the parameter values associated with each curve.

One word of explanation about the table is in order: an interesting case is the case in which the velocity dial error contains a "white noise" component. This can be obtained by letting $\alpha_2 \to \infty$ and $\frac{\beta_2}{\alpha_2} \to \mathcal{K}_2$ in Eq. (II.7). The exponential component of the velocity dial error then approaches white noise with spectral density $\mathcal{K}_2$ per cycle.

In the following table, the units for the various parameters are as follows:

γ_1 : (nautical miles)^2 (hours)$^{-4}$

γ_2 : (nautical miles)^2 (hours)$^{-2}$

γ_3 : (nautical miles)^2

γ_4 : (nautical miles)^2 (hours)$^{-2}$

α_2 : (hours)$^{-1}$

β_2 : (nautical miles)^2 (hours)$^{-2}$

The individual curves in Figs. 1-7 are identified by circled numbers. The parameter values associated with each curve are as follows:

Curve No.	γ_1	γ_2	γ_3	γ_4	α_2	β_2	$\lim \frac{\beta_2}{\alpha_2^2}$
1	50.0	1.0	.0010	9.0	$\to\infty$	$\to\infty$	.00010
2	50.0	1.0	.0010	∞	$\to\infty$	$\to\infty$	.0010
3	50.0	.10	.10	9.0	$\to\infty$	$\to\infty$	.0010
4	0.00	.10	.10	9.0	$\to\infty$	$\to\infty$	.0010
5	0.00	.10	.10	9.0	$\to\infty$	$\to\infty$	.10
6	10.0	1.0	.010	1.0	$\to\infty$	$\to\infty$	1.0
7	0.00	1.0	.10	1.0	$\to\infty$	$\to\infty$	.10
8	50.0	200	1.0	25.0	.60	2000	
9	50.0	200	1.0	25.0	.60	800	
10	50.0	200	1.0	25.0	1.80	800	
11	50.0	200	∞	∞	1.80	800	
12	50.0	200	9.0	400	1.80	2000	
13	50.0	200	9.0	400	.20	2000	
14	50.0	200	9.0	400	10.0	2000	
15	50.0	200	9.0	400	1000	2000	

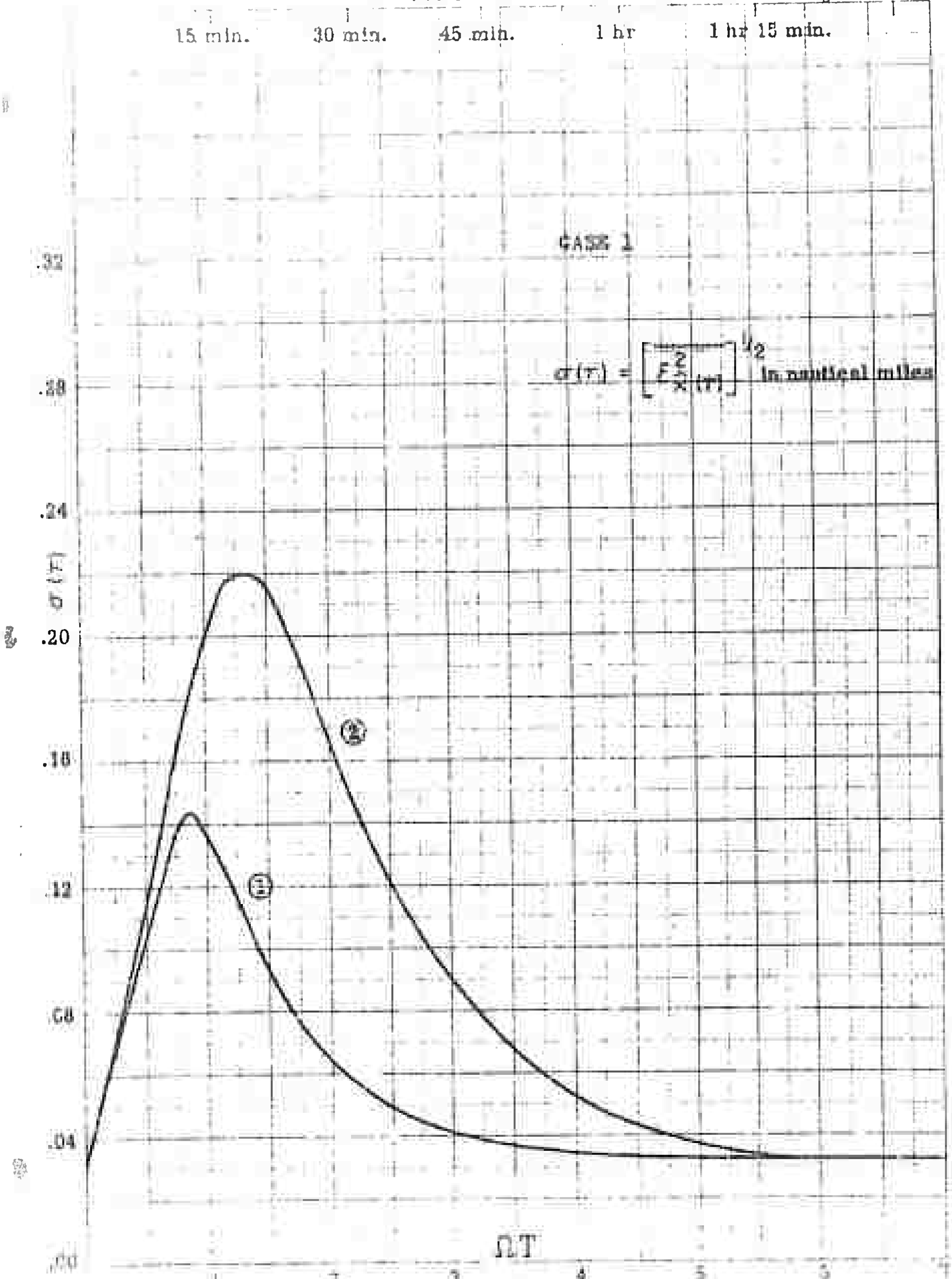

15 min.
30 min.
45 min.
1 hr
1 hr 15 min.
CASE 1
$\sigma(\tau) = \left[\overline{E_x^2(\tau)}\right]^{1/2}$ in nautical miles
.32
.28
.24
.20
.16
.12
.08
.04
.00
$\sigma(\tau)$
②
①
ΩT
1
2
3
4
5
6
7

Figure 1

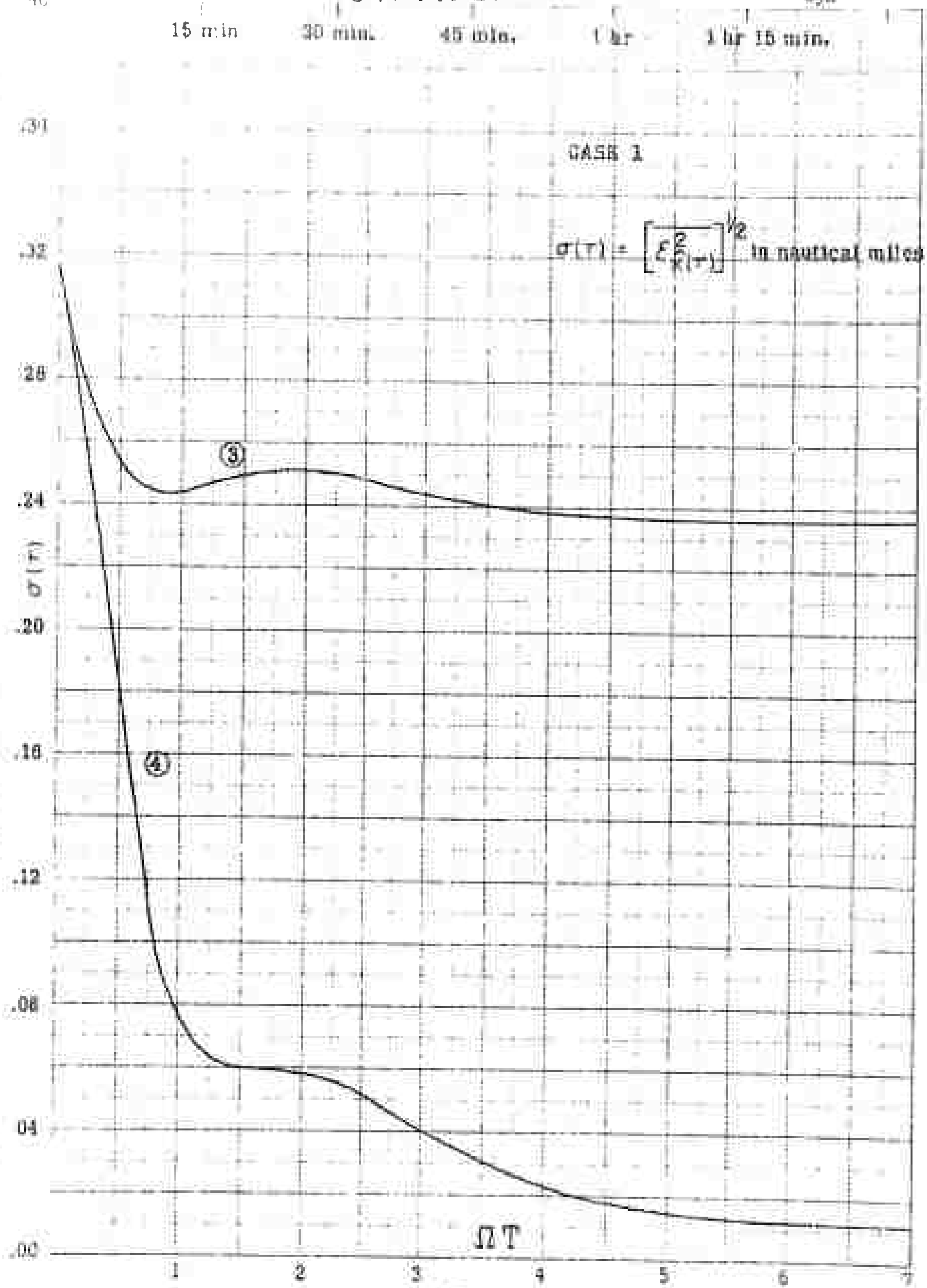

15 min
30 min.
45 min.
1 hr
1 hr 15 min.
CASE 1
$\sigma(\tau) = \left[\overline{\epsilon^2_{R(\tau)}}\right]^{1/2}$ in nautical miles
$\sigma(\tau)$
.32
.28
.24
.20
.16
.12
.08
.04
.00
3
4
1
2
3
4
5
6
7
ΩT

Figure 2

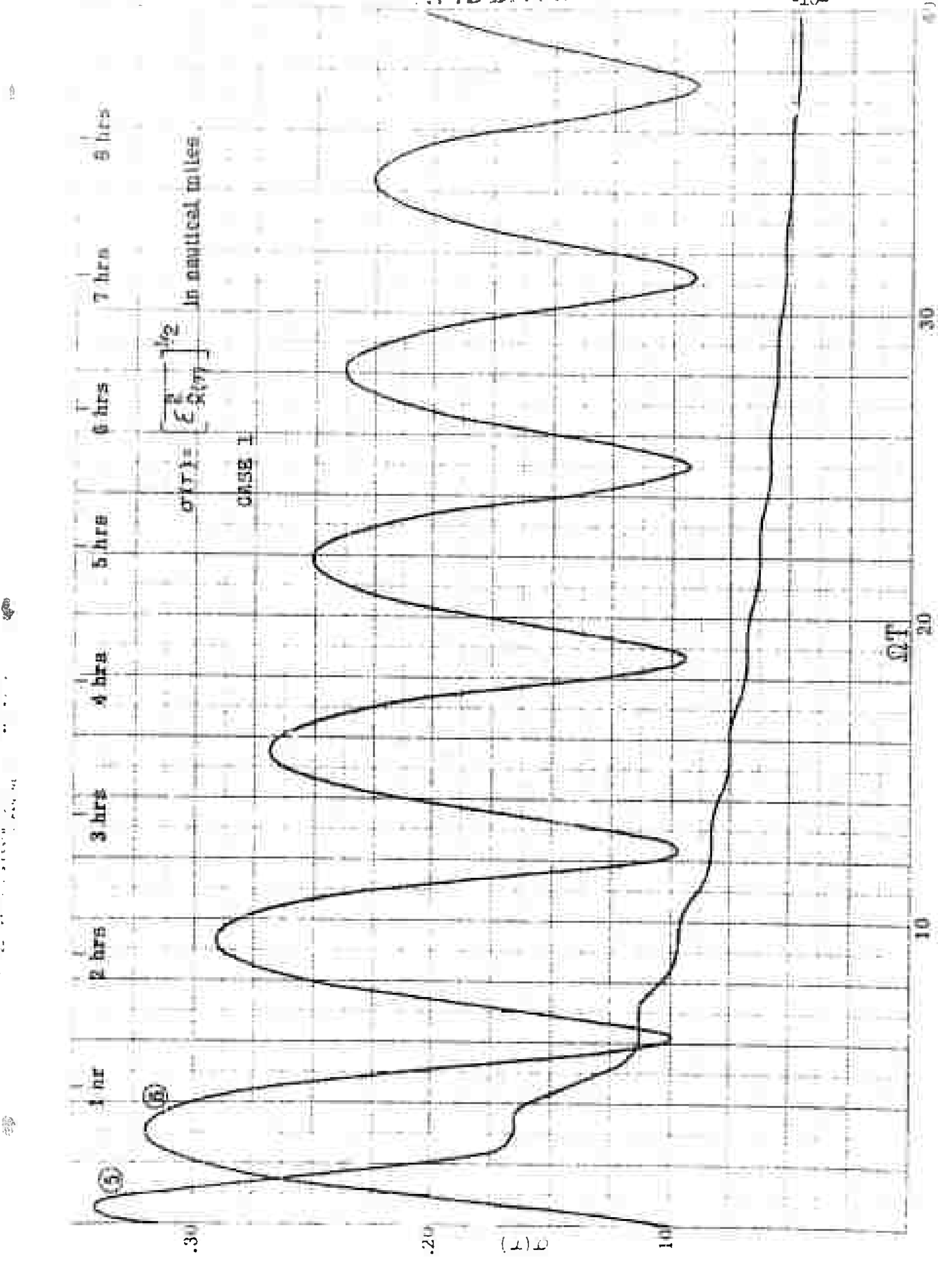

1 hr
2 hrs
3 hrs
4 hrs
5 hrs
6 hrs
7 hrs
8 hrs
$\sigma(T) = \left[\frac{\varepsilon^2}{R(T)}\right]^{1/2}$ in nautical miles
CASE I
5
6
.30
.20
.10
$\sigma(T)$
10
20
30
40
ΩT

Figure 3

CONFIDENTIAL

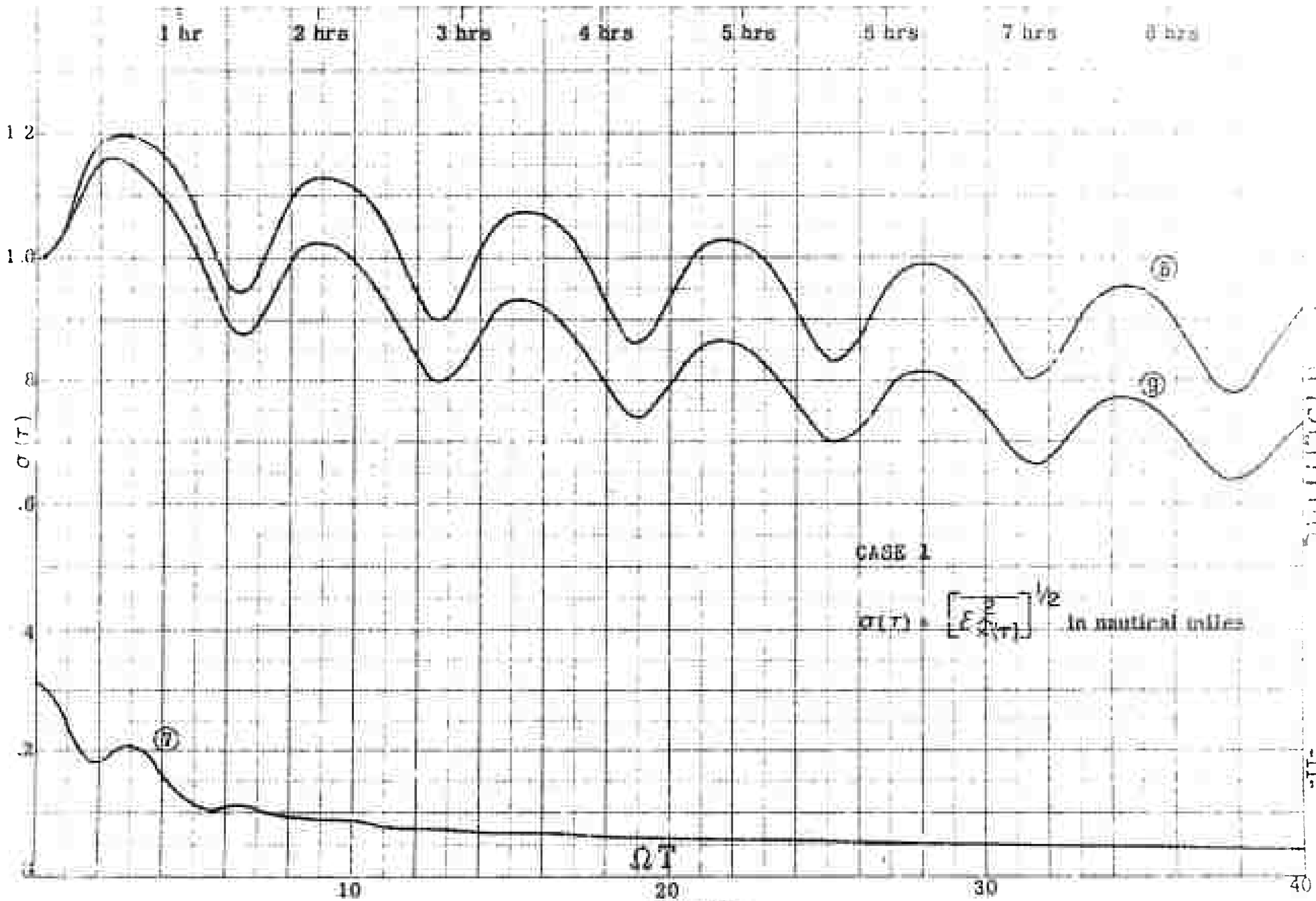


Figure 4

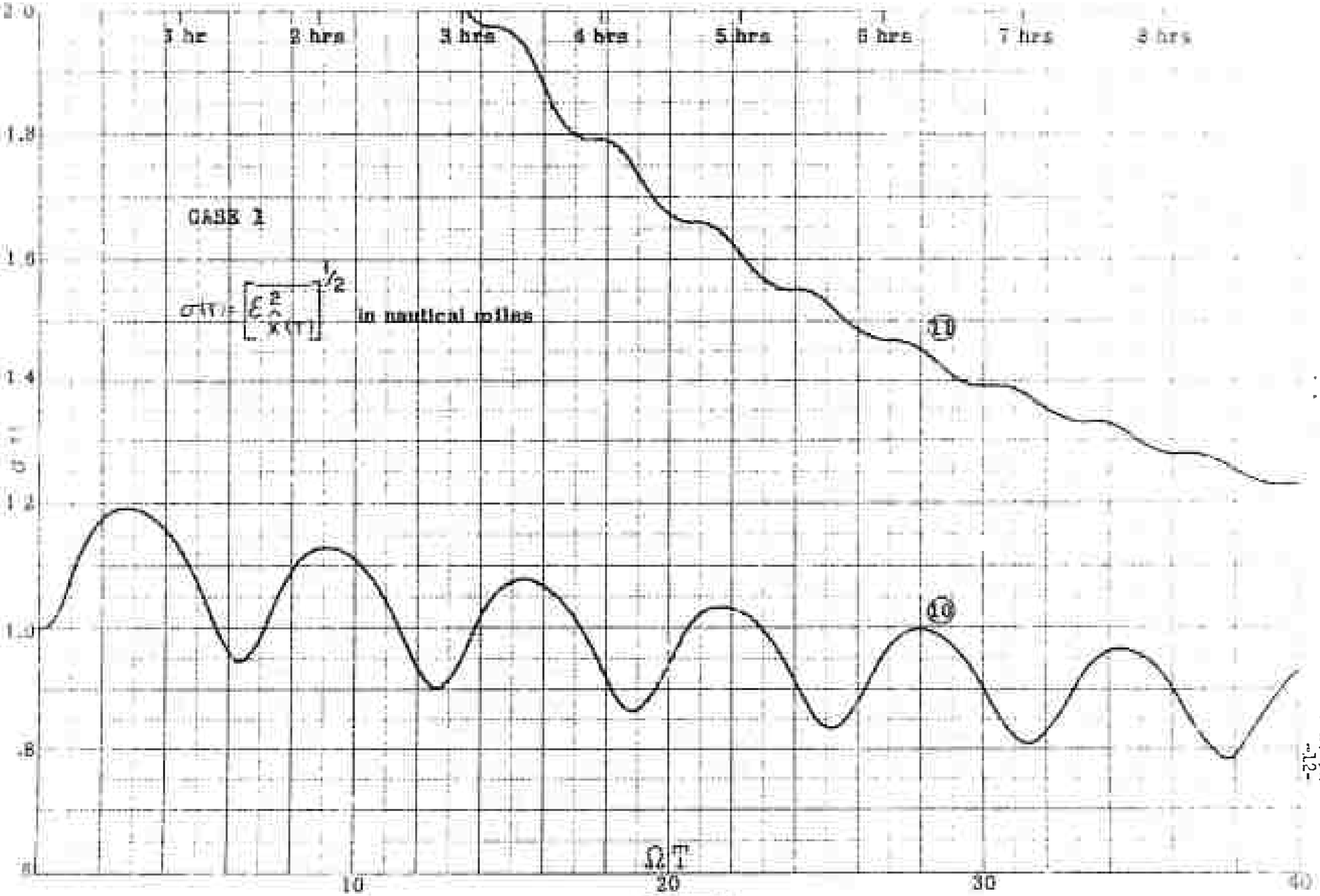
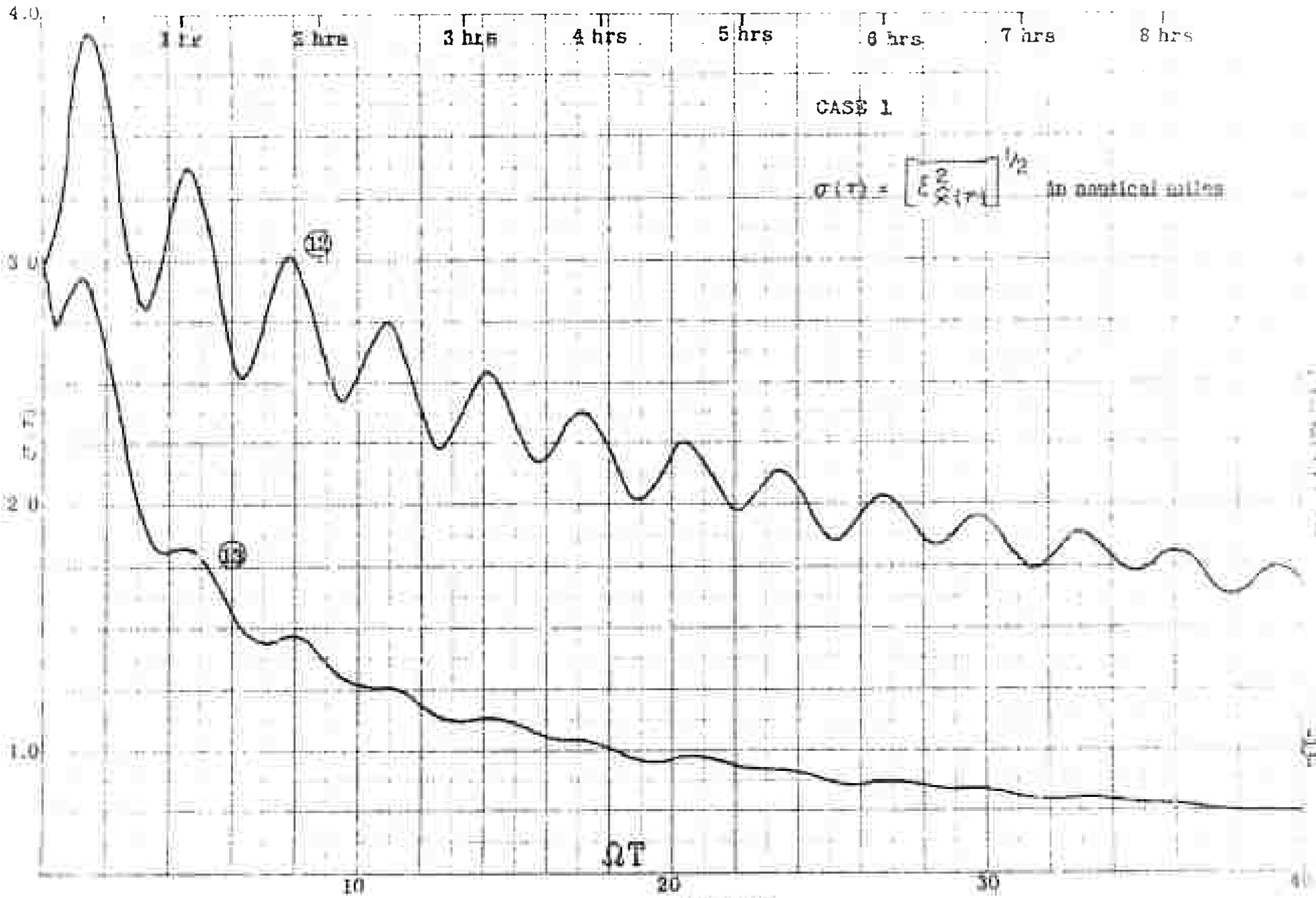
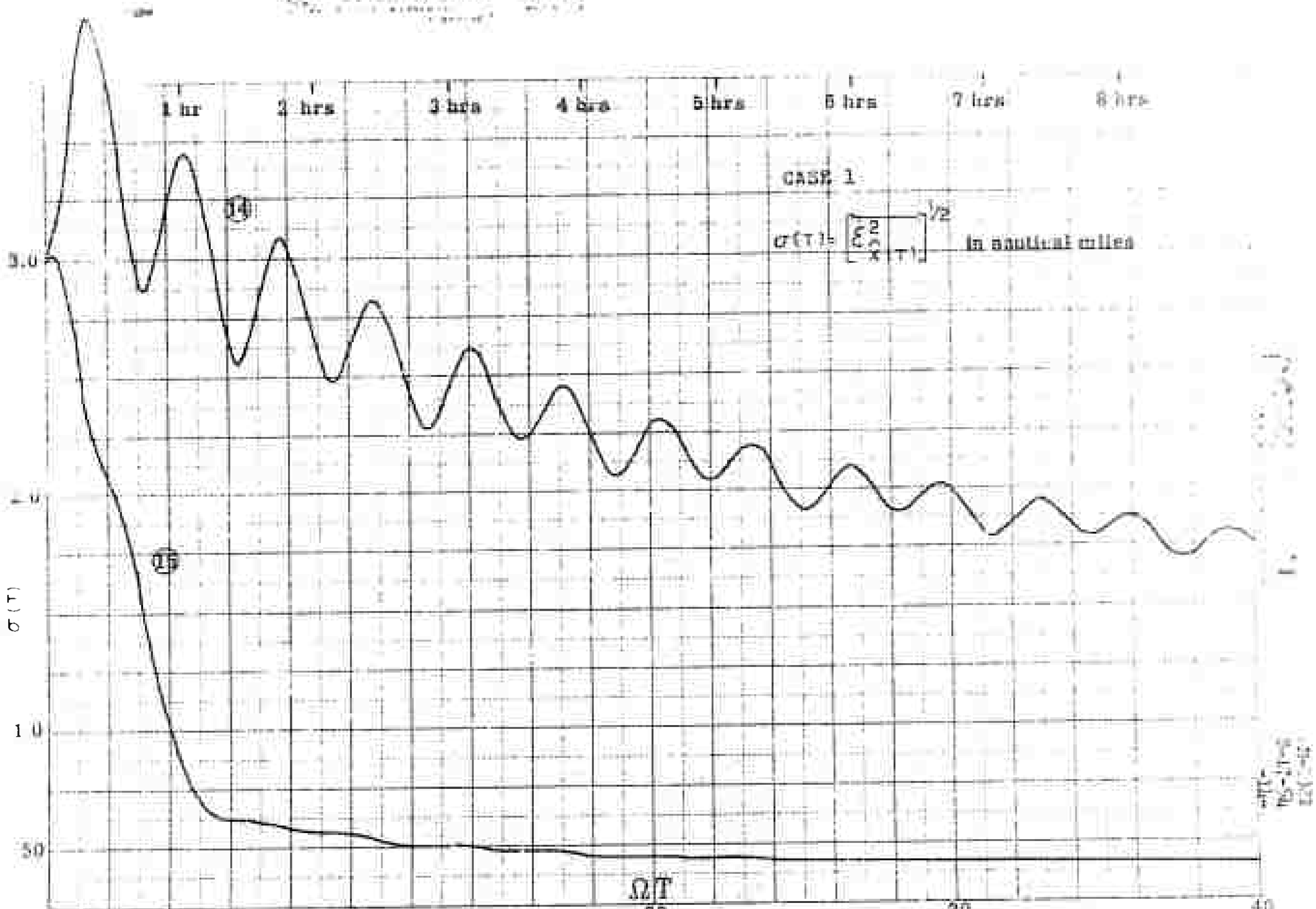

CASE I
$\sigma(T) = \left[\mathcal{E}\,\hat{x}^2(T)\right]^{1/2}$ in nautical miles
1 hr
2 hrs
3 hrs
4 hrs
5 hrs
6 hrs
7 hrs
8 hrs
2.0
1.8
1.6
1.4
1.2
1.0
.8
0
10
20
30
40
ΩT
11
10

Figure 5



1 hr
2 hrs
3 hrs
4 hrs
5 hrs
6 hrs
7 hrs
8 hrs
CASE 1
$\sigma(\tau) = \left[\overline{\varepsilon^2_{x(\tau)}}\right]^{1/2}$ in nautical miles
4.0
3.0
2.0
1.0
12
13
ΩT
10
20
30
40

Figure 6



1 hr
2 hrs
3 hrs
4 hrs
5 hrs
6 hrs
7 hrs
8 hrs
CASE 1
$\sigma(T) = \left[\overline{\epsilon_x^2(T)}\right]^{1/2}$ in nautical miles
14
15
3.0
1.0
.50
$\sigma(T)$
10
20
30
40
ΩT

Figure 7

III. CASE 2: $\phi_V(s,t) = \gamma_2$

As in Case 1, let $0 = t_o < t_1 \cdots < t_{n-1} = T$ be n equally spaced time points in $(0,T)$.

Let

$$\text{(III.1)} \qquad \left[\eta_{ij}\right] = \left[\phi_B(t_i,\ t_j)\right]^{-1} \qquad \text{(matrix inverse)}$$

Also let

$$\left[A_{11}\right]_n = \Omega^4 \sum_{i,j=o}^{n-1} \eta_{ij}$$

$$\text{(III.2)} \qquad \left[A_{12}\right]_n = \Omega^4 \sum_{i,j=o}^{n-1} \eta_{ij}\, t_i$$

$$\left[A_{22}\right]_n = \Omega^4 \sum_{i,j=o}^{n-1} \eta_{ij}\, t_i\, t_j$$

and

$$A_{11} = \lim_{n \to \infty} \left[A_{11}\right]_n$$

$$\text{(III.3)} \qquad A_{12} = \lim_{n \to \infty} \left[A_{12}\right]_n$$

$$A_{22} = \lim_{n \to \infty} \left[A_{22}\right]_n$$

The quantities A_{11}, A_{12}, A_{22} are functions of T.

Reference 1 gave formulas (eqs. IV.14, IV.15, IV.17 of Ref. 1) from which could be derived the formula for $\overline{\epsilon_{\hat{x}}^2(t;T)}$. The result can be expressed as follows:

Let

$$\text{(III.4)}\quad \begin{cases} K_{11} = A_{11} + \dfrac{1}{\gamma_3} \\ K_{12} = A_{12} \\ K_{22} = A_{22} + \dfrac{1}{\gamma_2} + \dfrac{1}{\gamma_4} \\ \Delta = K_{11} K_{22} - K_{12}^2 \end{cases}$$

Then, for $0 \leqq t \leqq T$,

$$\text{(III.5)}\quad \overline{\epsilon_{\hat{x}}^2(t;T)} = \frac{1}{\Delta}\left\{ K_{22} - 2K_{12}\,t + K_{11}\,t^2 \right\}$$

and therefore

$$\text{(III.5')}\quad \overline{\epsilon_{\hat{x}(T)}^2} = \frac{1}{\Delta}\left\{ K_{22} - 2\,K_{12}\,T + K_{11}\,T^2 \right\}$$

These formulas hold for any $\phi_B(s,t)$.

In Ref. 1, explicit expressions were obtained for A_{11}, A_{12}, A_{22} for the case

$$\text{(III.6)}\quad \phi_B(s,t) = \gamma_1 + \beta_1 e^{-\alpha_1 |s-t|}$$

The results were

$$A_{11} = \frac{\Omega^4(2+\alpha_1 T)}{2\beta_1}\left[1 - \frac{1}{1 + \frac{2\beta_1}{\gamma_1(2+\alpha_1 T)}}\right] \tag{III.7i}$$

$$A_{12} = \frac{\Omega^4(2+\alpha_1 T)\,T}{4\beta_1}\left[1 - \frac{1}{1 + \frac{2\beta_1}{\gamma_1(2+\alpha_1 T)}}\right] \tag{III.7ii}$$

$$A_{22} = \frac{\Omega^4 T^2}{2\beta_1}\left[1 + \frac{\alpha_1 T}{3} + \frac{1}{\alpha_1 T} - \frac{\frac{1}{4}(2+\alpha_1 T)}{1 + \frac{2\beta_1}{\gamma_1(2+\alpha_1 T)}}\right] \tag{III.7iii}$$

In Figs. 8-10 are shown curves of $\left[\overline{\varepsilon^2_{\hat{x}(T)}}\right]^{1/2}$ as a function of T for $\phi_B(s,t) = \gamma_1 + \beta_1 e^{-\alpha_1|s-t|}$. Each curve is determined by specification of the six statistical parameters γ_1, γ_2, γ_3, γ_4, α_1, β_1.

An interesting case is the case in which the accelerometer dial error contains a "white noise" component. This can be obtained by letting $\alpha_1 \rightarrow \infty$ and $\frac{\beta_1}{\alpha_1} \rightarrow \mathcal{K}_1$ in Eq. (III.6). The exponential component of the accelerometer dial error then approaches white noise with spectral density $\mathcal{K}_1$ per cycle.

In the following table, the units for the various parameters are as follows:

γ_1 : (nautical miles)^2 (hours)$^{-4}$

γ_2 : (nautical miles)^2 (hours)$^{-2}$

γ_3 : (nautical miles)^2

γ_4 : (nautical miles)^2 (hours)$^{-2}$

α_1 : (hours)$^{-1}$

β_1 : (nautical miles)^2 (hours)$^{-4}$

The parameter values associated with each curve in Figs. 8-10 are as follows:

Curve No.	γ_1	γ_2	γ_3	γ_4	α_1	β_1	$\lim \frac{\beta_1}{\alpha_1}$
1	50.0	1.0	.30	1.0	$\to \infty$	$\to \infty$	.20
2	0.00	1.0	.10	9.0	$\to \infty$	$\to \infty$	.20
3	0.00	1.0	.10	9[illegible]	10	800	
4	50.0	1.0	.10	1.0	10	100	
5	25.0	1000	1.0	25.0	10	25.0	
6	0.00	1000	1.0	25.0	10	50.0	

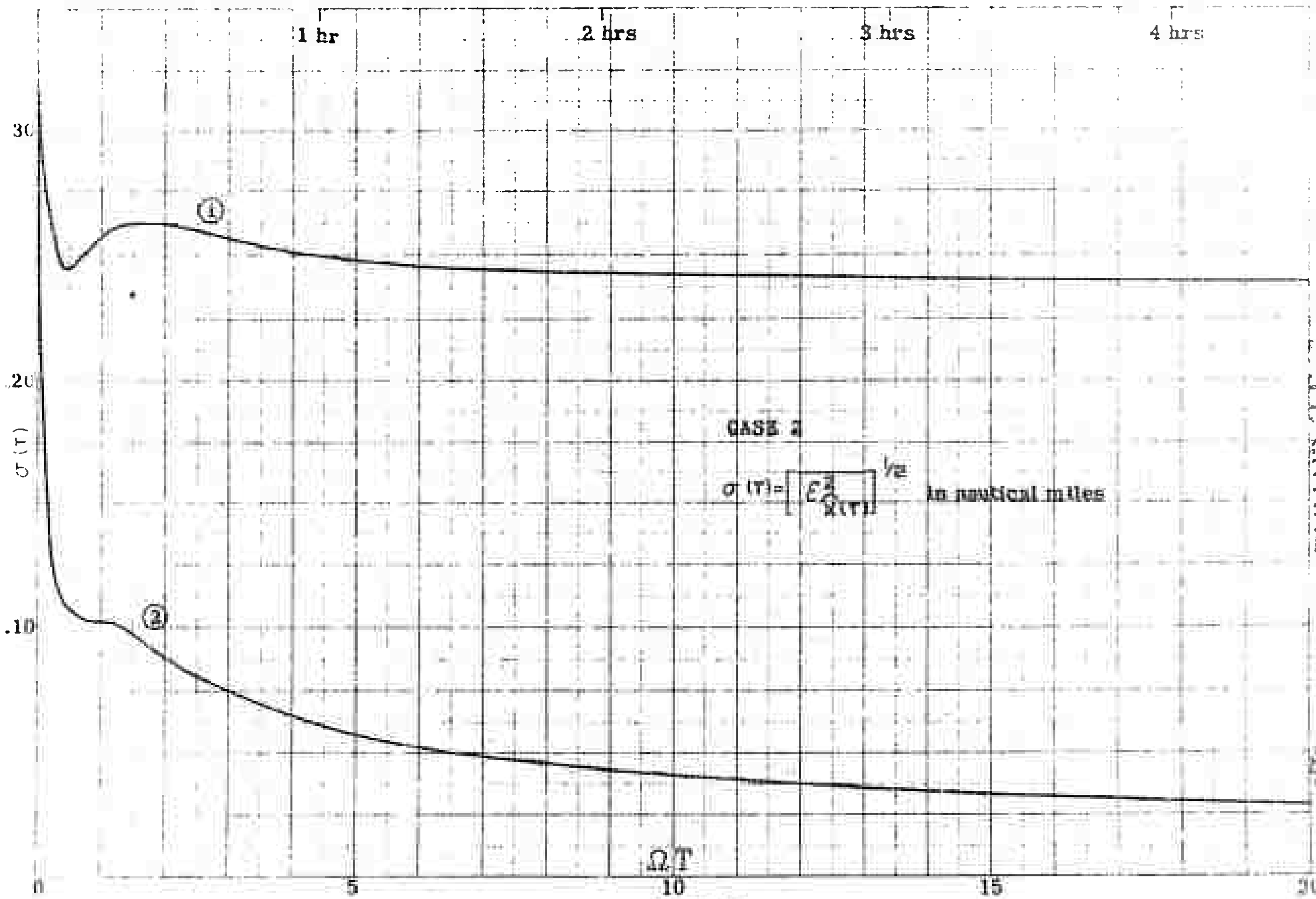

1 hr
2 hrs
3 hrs
4 hrs
.30
.20
.10
$\sigma(T)$
CASE 2
$\sigma(T) = \left[\boxed{\mathcal{E}^2_{\Omega(T)}} \right]^{1/2}$ in nautical miles
0
5
10
15
20
ΩT

Figure 8

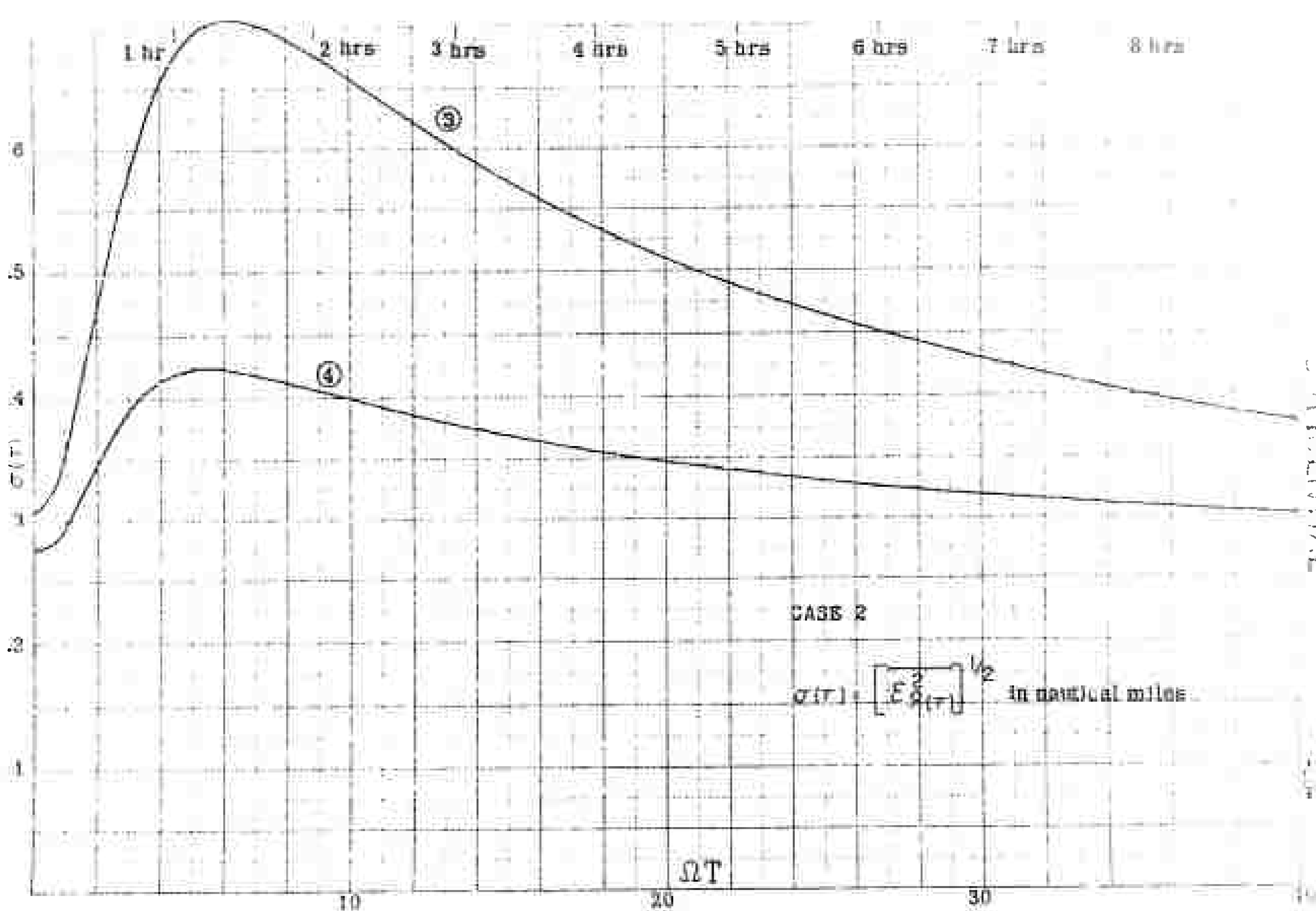

1 hr
2 hrs
3 hrs
4 hrs
5 hrs
6 hrs
7 hrs
8 hrs
3
4
.6
.5
.4
.3
.2
.1
CASE 2
$\sigma(T) = \left[\overline{\mathcal{E}^2_{(T)}}\right]^{1/2}$ in nautical miles
10
20
30
ΩT

Figure 9

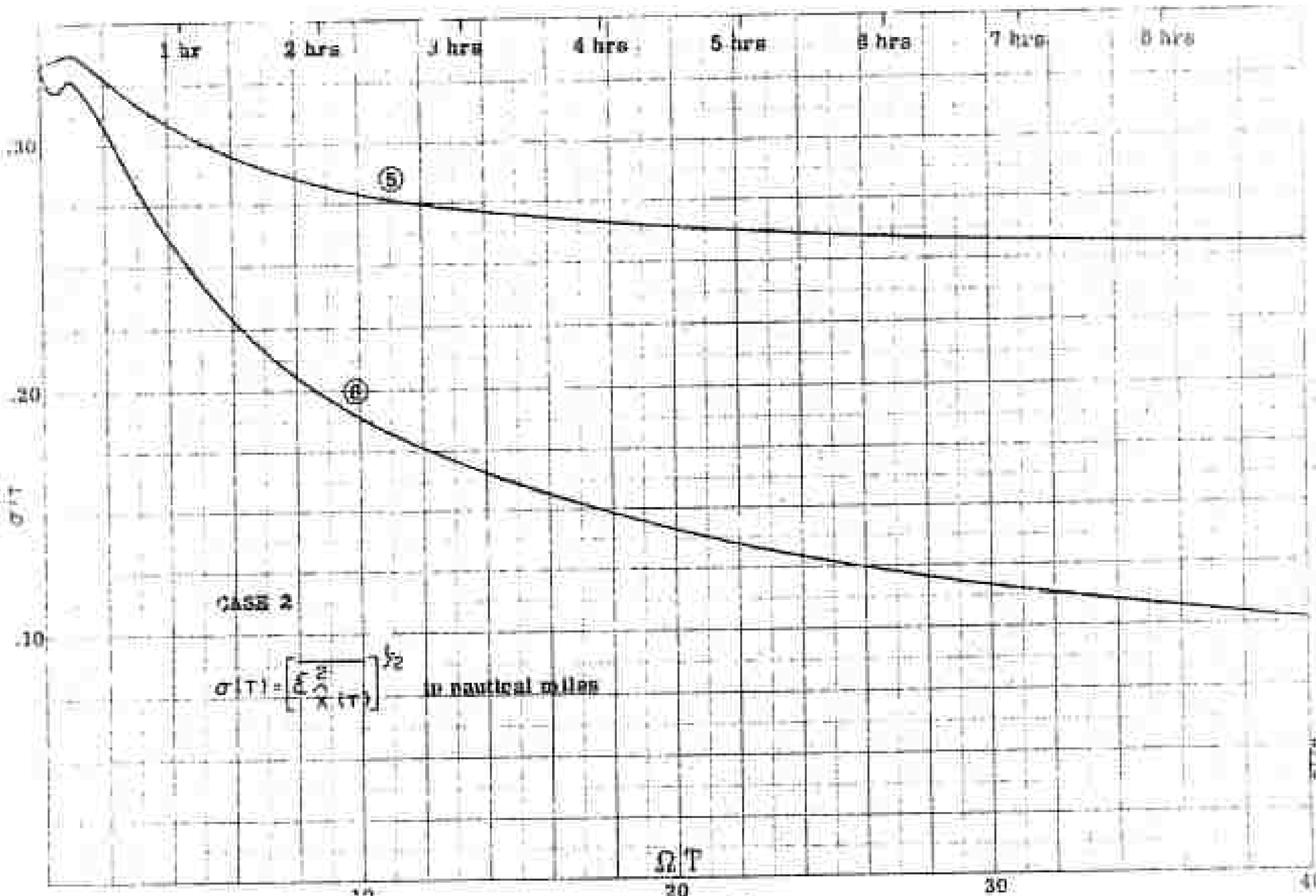

1 hr
2 hrs
3 hrs
4 hrs
5 hrs
6 hrs
7 hrs
8 hrs
.30
.20
.10
σ(T)
(5)
(6)
CASE 2
$\sigma(T) = \left[\overline{\epsilon^2_{\hat{x}}(T)}\right]^{1/2}$ in nautical miles
10
20
30
40
ΩT

Figure 10

REFERENCES

1. Swerling, P. and E. Reich, Statistical Theory of Navigation Employing Independent Inertial and Velocity Measurements, The RAND Corporation, Research Memorandum RM-1220, March 25, 1954. (Confidential)